PEANO ARITHMETIC USING LAMBDA CALCULUS

WHAT IS $\mathbb{N}$?

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inductive Nat where
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    | succ : Nat → Nat
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According to Haskell:

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data Nat where
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- $\mathbb{N}$ is the initial object of Mod_F.

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For $\mathbb{N}$:

- $0 : \mathrm{pt} \to \mathbb{N}$ picks out 0,
- $+1 : \mathbb{N} \to \mathbb{N}$ is the successor function.

Observe that, in **Set**, the initial object of the endofunctor $f : X \to 1 + X$ is $\langle 0, ++ \rangle : 1 + \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ contains 0 the smallest natural and $++$ is the successor operator. Because monomorphisms to initial objects are isomorphisms, we can conclude that the only subalgebra of $\mathbb{N}$ is $\mathbb{N}$ itself.
This is the principle of induction.

CRASH COURSE
ON
LAMBDA CALCULUS

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Function application is done by (left associative) juxtaposition:

- "$f\ x$" is "$f(x)$",
- "$(\lambda a.\, a\ b)\ g$" is "$g\ b$" is "$g(b)$".

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```

HOW DO WE COUNT IN LAMBDA CALCULUS?

CHURCH ENCODING

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In lambda calculus:

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ARITHMETIC

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pred ≔ ???
```

How do we subtract 1??!?

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Wikipedia

SCOTT ENCODING

THE OTHER SCHOOL OF THOUGHT

Inductive types have this great feature called

✨*pattern matching*✨

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data Tree a = Leaf
            | Node (Tree a) a (Tree a)
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We can also do pattern-matching for `Nat`!

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- the input f is a “continuation” of the computation.

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- the Yoneda embedding $A \mapsto h_A = \mathrm{Hom}(A, -)$…

Wait 🤔, maybe all of these examples are the last example…


Always has been
Wait, it's all Yoneda?

Taking a closer look 🤔

$$\mathrm{ev}_a : \forall R.\,(A \to R) \to R, \quad \mathrm{ev}_a(f) = f(a)$$

ev_a is just a natural transformation $\mathrm{Hom}(A, -) \Rightarrow \mathrm{Id}$!!!

By Yoneda, $A \cong \mathrm{Nat}(\mathrm{Hom}(A, -), \mathrm{Id})$.

SLOGAN

PATTERN MATCHING + YONEDA = SCOTT ENCODING

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So for a tree $t \in T_A$, evaluation at t looks like

$$\mathrm{ev}_t : \forall R.\, R \to (T_A \times A \times T_A \to R) \to R.$$

THE SCOTT TREE

```
type STree a
    =  forall r.
       r                                    -- Leaf                        -> r
    -> (STree a -> a -> STree a -> r) -- Node (Tree a) a (Tree) -> r
    -> r
```

A tree is not so dissimilar from a natural number…

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data Nat = Zero
         | Succ Nat
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- $n = 0$: the zero case,
- $n = n' + 1$: the successor case.

The $\mathbb{N} \to R$ "deconstructs" the successor case into the predecessor:

```
pred :: Nat -> Nat
pred n = case n of
    Zero    -> undefined
    Succ n' -> n' -- here
```

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```
type SNat = forall r. r -> (SNat -> r) -> r

pred :: SNat -> SNat
pred n = n undefined (\n' -> n')
```

Dead simple!

Okay, but we still need to solve the Church version.

$$\forall R.\, R \to (\mathbb{N} \to R) \to R$$

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SO WHAT'S GOING ON?

REVISITING *F*-MODULES

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Given an endofunctor $F : \mathrm{Sets} \to \mathrm{Sets}$, we can talk about its module category $\mathrm{Mod}_F := \mathrm{Mod}_F(\mathrm{Sets})$ defined as before.

$$(\alpha : F(X) \longrightarrow X)$$

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$$(\alpha : F(X) \longrightarrow X)$$

If Mod_F has an initial object $(\mathbb{M}, i)$, the unique map

$$! : (\mathbb{M}, i) \to (X, \alpha)$$

is called the *fold* map or *catamorphism*.

For example, for a fixed set A, the endofunctor

$$F(X) = \mathrm{pt} \sqcup (A \times X)$$

has as the initial object of its module category List(A):

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data List a = [] | a : (List a)
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Aside: The colon ":" is used to denote concatenation onto a list. So for example

$$a_0{:}a_1{:}a_2{:}[] = a_0a_1{:}[a_2] = a_0{:}[a_1, a_2] = [a_0, a_1, a_2].$$

$$F(X) = \mathrm{pt} \sqcup (A \times X)$$

The map $\alpha : F(X) \to X$ of an F-module consists of:

- a map $e : \mathrm{pt} \to X$, or just $e \in X$; and
- a map $c : A \times X \to X$.

The *fold* map out of List(A) is so called because the unique morphism $! : \text{List}(A) \to (X, \alpha)$ must satisfy (proof)

- $!([]) = e$,
- $!(a{:}as) = c(a, !(as))$.

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So the fold of a list $[a_0, a_1, a_2, \cdots, a_n]$ is forced to be

$$!([a_0, a_1, a_2, \ldots, a_n]) = c(a_0, c(a_1, c(a_2, c(\ldots, c(a_n, e))))).$$

$!$ recursively "folds" the list into a single element of X.

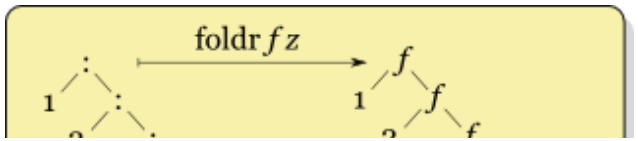
Fold (higher-order function)

In functional programming, a **fold** is a higher-order function that analyzes a recursive data structure and, through use of a given combining operation, recombines the results of recursively processing its constituent parts, building up a return value. Fold is also termed as **reduce**, **accumulate**, **aggregate**, **compress**, or **inject**. Typically, a fold is presented with a combining function, a top node of a data structure, and possibly some default values to be used under certain conditions. The fold then proceeds to combine elements of the data structure's hierarchy, using the function in a systematic way.

Folds are in a sense dual to unfolds, which take a *seed* value and apply a function corecursively to decide how to progressively construct a corecursive data structure, whereas a fold recursively breaks that structure down, replacing it with the results of applying a combining function at each node on its terminal values and the recursive results (catamorphism, versus anamorphism of unfolds).

As structural transformations

Folds can be regarded as consistently replacing the structural components of a data structure with functions and values. Lists, for example, are built up in many functional languages from two primitives: any list is either an empty list, commonly called *nil* (`[]`), or is constructed by prefixing an element in front of another list, creating what is called a *cons* node (`Cons(X1,Cons(X2,Cons(...(Cons(Xn,nil)))))`), resulting from application of a `cons` function (written down as a colon `(:)` in Haskell). One can view a fold on lists as *replacing* the *nil* at the end of the list with a specific value, and *replacing* each *cons* with a specific function. These replacements can be viewed as a diagram:



For natural numbers, the endofunctor is

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(A special case of lists $F(X) = \mathrm{pt} \sqcup (A \times X)$ with $A = \mathrm{pt}$.)

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The *fold* map $! : \mathbb{N} \to (X, \alpha)$ is thus $!(n) = s^n(z)$: precisely the Church encoding (proof).

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&\cong \int_{(X,\alpha)\in \mathrm{Mod}_F} X.
\end{aligned}$$

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But fixing $X \in \mathrm{Sets}$, F-module structures on X are simply elements of $\mathrm{Sets}(F(X), X) = \mathrm{Sets}(\mathrm{pt} \sqcup X, X) = X \times \mathrm{Sets}(X, X)$.

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Therefore

$$\begin{aligned} \mathbb{N} &\cong \int_{(X,\alpha) \in \mathrm{Mod}_F} X \\ &\cong \int_{X \in \mathrm{Sets}} \mathrm{Sets}(\mathrm{Sets}(F(X), X), X) \end{aligned}$$

But fixing $X \in \mathrm{Sets}$, F-module structures on X are simply elements of $\mathrm{Sets}(F(X), X) = \mathrm{Sets}(\mathrm{pt} \sqcup X, X) = X \times \mathrm{Sets}(X, X)$.

Therefore

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\mathbb{N} &\cong \int_{(X,\alpha) \in \mathrm{Mod}_F} X \\
&\cong \int_{X \in \mathrm{Sets}} \mathrm{Sets}(\mathrm{Sets}(F(X), X), X) \\
&\cong \int_{X \in \mathrm{Sets}} \mathrm{Sets}(\mathrm{Sets}(X \times \mathrm{Sets}(X, X), X), X).
\end{aligned}$$

$$\mathbb{N} \cong \int_{X \in \mathsf{Sets}} \mathsf{Sets}(X \times \mathsf{Sets}(X, X), X).$$

What this says is that natural numbers are exactly (natural) families of functions

$$X \to (X \to X) \to X.$$

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This now tells us that natural numbers are also (natural) families of functions

$$X \to (\mathbb{N} \to X) \to X.$$

WHAT IS $\mathbb{N}$?

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Church encoding:

$$\mathbb{N} \cong \int_{X \in \mathrm{Sets}} \mathrm{Sets}(\mathrm{Sets}(F(X), X), X)$$

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- Receives result of recursion.

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- Does one level of pattern matching.

WHAT DOES LEAN DO?

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- Objects are morphisms $\alpha : F(\mathbb{M} \times X) \to X$ in Sets.
- Morphisms are the obvious commutative diagrams.
- Repeating the Yoneda argument,

$$\mathbb{N} \cong \int_{X \in \mathrm{Sets}} \mathrm{Sets}(\mathrm{Sets}(F(\mathbb{N} \times X), X), X).$$

We can see this in the recursor for the natural numbers here:

```
Nat.rec.{u} {motive : Nat → Sort u}
  (zero : motive zero)
  (succ : (n : Nat) → motive n → motive n.succ)
  (t : Nat) :
  motive t
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Let's reduce to the non-dependent setting by choosing a constant motive motive(n) = X.

After providing `t : Nat`, the signature reduces to

$$X \to (\mathbb{N} \to X \to X) \to X.$$

`pred`

Let’s solve our original problem now.

IDEA

$$n\ f\ x = \underbrace{f\ (f\ (\cdots\ (f}_{n \text{ times}}\ x)))$$

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$$\text{pred}\ n\ f\ x = n\ f'\ x' = \underbrace{f'\ (f'\ (\cdots (f'}_{n \text{ times}}\ x')))$$

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$$f'\ (\text{true},\ x) = (\text{false},\ x)$$
$$f'\ (\text{false}, x) = (\text{false}, f\ x)$$

$$\text{pred}\, n\, f\, x = n\, f'\, x' = \text{pr}_2\ \underbrace{f'\,(f'\,(\cdots(f'}_{n \text{ times}}\ x')))$$

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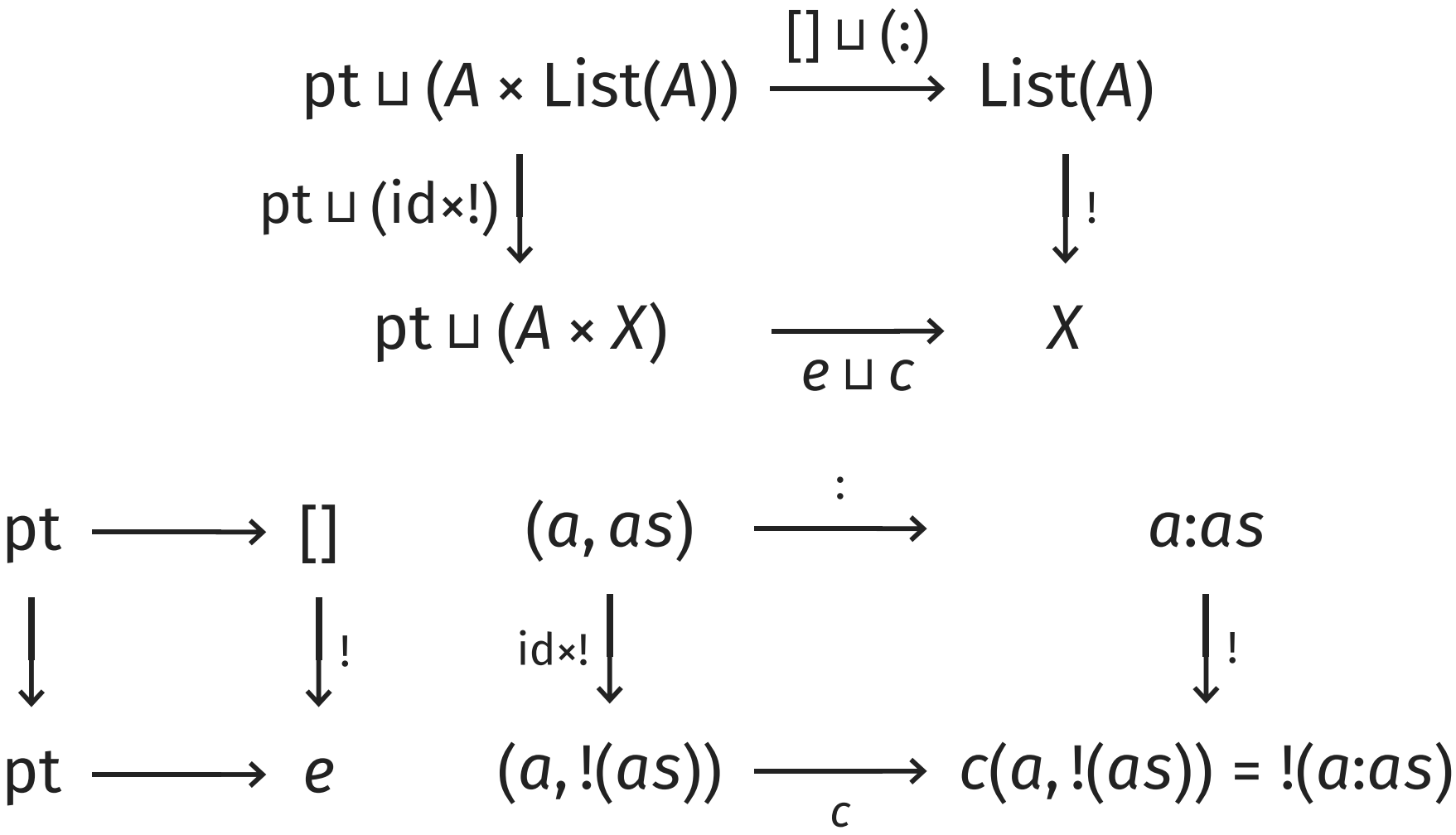
…but in lambda calculus.

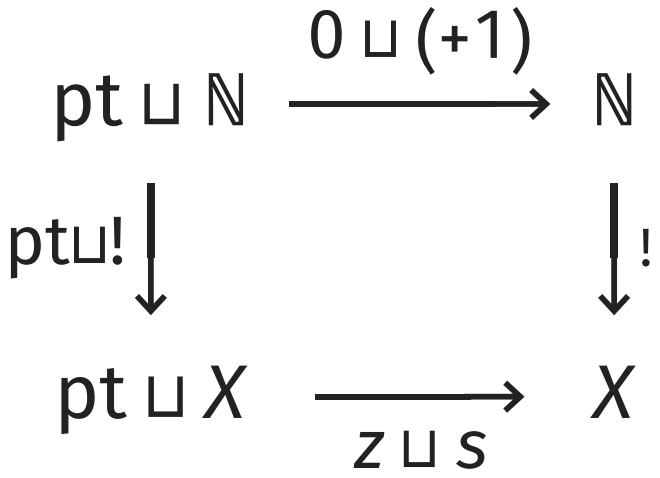
We use Church/Scott encoding!

$$|-| : \mathrm{Fin} \to \mathbb{N}$$

APPENDIX



Go back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ℕ
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X
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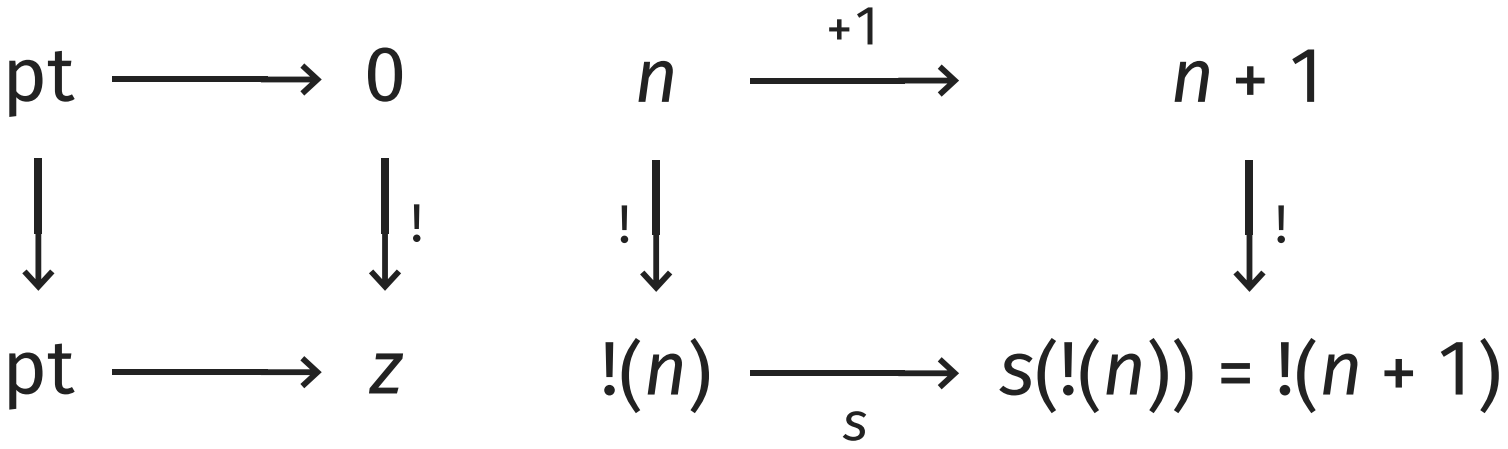
+1
pt
0
n
n + 1
!
!
!
pt
z
!(n)
s(!(n)) = !(n + 1)
s